UCLA/90/TEP/72

Quantum gravitational measure for three-geometries*

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ABSTRACT

The gravitational measure on an arbitrary topological three-manifold is constructed. The nontrivial dependence of the measure on the conformal factor is discussed. We show that only in the case of a compact manifold with boundary the measure acquires a nontrivial dependence on the conformal factor which is given by the Liouville action. A nontrivial Jacobian (the divergent part of it) generates the Einstein-Hilbert action. The Hartle-Hawking wave function of Universe is given in terms of the Liouville action. In the gaussian approximation to the Wheeler-DeWitt equation this result was earlier derived by Banks et al. Possible connection with the Chern-Simons gravity is also discussed.

⋆ This work was supported in part by Department of Energy grant DE-AT03-88ER 40384
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General covariance plays the central role in physics. Quantization of systems with a large symmetry (general covariance, gauge invariance) requires proper geometrical tools. Indeed, the Feynman functional integral approach has proven extremely useful in formulating properly the quantum dynamics of random paths (a point particle) and surfaces (strings, two-geometries) [1,2,3]. The problem of great interest now is how to extend the Feynman and the Polyakov approach to the case of quantum geometries ($d > 2$) [4]. It seems that we require the knowledge of how to sum over random three- and four-geometries in order to approach the problem of quantum gravity. The simpler case of two-geometries was understood in recent several years [2,3,5,]. Recently the subject of non-critical strings or induced quantum 2d gravity with its Liouville theory formulation has received great attention [5]. Also, recent interest in topology change and wormholes (topology changing amplitudes in quantum gravity) seems to indicate that the rigorous proper gravitational measure on manifolds of an arbitrary topology is required in order to make discussion of these issues more quantitative.

In this Letter we study for simplicity the gravitational measure for three-geometries. This is an example of the general result for the quantum gravitational measure in any dimension. We extend the Polyakov approach to the case of three-geometries and show some qualitative differences. In particular, the absence of conformal anomalies in three dimensions (on three-geometries without boundary) makes a difference. However, the nontrivial Jacobian in the measure must be regularized and as a result we must add the bare local Einstein-Hilbert action as a local counterterm to the effective action. The local Einstein-Hilbert action is generated by the Jacobian in the measure. It seems that all infinities generated by the measure can be taken care off by simple renormalization of the cosmological and Newton constant in the Einstein-Hilbert action. In this sense we expect that the theory satisfy the criterium of calculability.

The subject of the rest of this Letter is the geometrical construction of gravitational measure in any dimension d and calculation of the induced effective action for the conformal factor for three-geometries.

Consider the space of metrics $Riem(M) = Q(M)$ on a given manifold M. A tangent space $TQ_{|g}$ at a metric g is spanned by the vectors δg. The volume form on Q can be introduced (almost) uniquely after Q is equipped with the riemannian structure, i. e., a quadratic form $<,>_{TQ}$ on the tangent space TQ. It is the property of the Riemannian geometry that a metric defines the unique volume form $\sqrt{g}d^n x$. This is the "square-root of determinant of the metric" rule which determines the measure (volume form). The requirement of ultralocality of the measure leads to (an almost) unique metric on Q. This is the well known DeWitt metric on $Riem(M)$ [7] utilized by Polyakov in his work on quantum (random) two-geometries [2]

$$G^{abcd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}) + Cg^{ab}g^{cd}, \tag{1}$$

where $C > -\frac{2}{d}$. With this metric one defines the norm or scalar product on TQ

$$||\delta g||^2 =< \delta g, \delta g >= \int dx\sqrt{g}G^{abcd}\delta g_{ab}\delta g_{cd}. \tag{2}$$

The Polyakov gaussian measure is defined by the condition

$$\int d\mu(\delta g)e^{-\frac{1}{2}||\delta g||^2} = 1. \tag{3}$$

The space of metrics Q has locally a form of the fiber bundle $Q = S \times Diff$, where $S = Q/Diff$ is the Wheeler-DeWitt superspace defined as a space of metrics modulo diffeomorphisms. At some "points" g_0 of Q which are fixed "points" of a subgroup of $Diff$ which is the isometry group of g_0, $Isom(g_0)$, S is not a manifold but rather an infinite dimensional version of an orbifold [6]. In order to define a measure on S we need to divide out the invariant measure on the fiber, the gauge group $Diff(M)$ of general covariance. We will proceed by introducing the natural co-ordinate system a part of which are group coordinates on $Diff(M)$. The space of metrics on a manifold M, $Q(M) = Riem(M)$ is a space of maps from M into the coset space $GL^+(d, R)/SO(d)$. Utilizing the decomposition of the

coset $GL^+(d,R)/SO(d) = R^+ \times SL(d,R)/SO(d)$ we notice that the R^+ factor corresponds to an arbitrary positive scale of a metric which can be changed by the action of two groups of transformations. One of them is the group of diffeomorphisms $Diff_0(M)$ and the other is the Weyl conformal rescaling group $Weyl(M)$. Consider a conformal equivalence class of metrics on M, $g \equiv \bar{g}$ iff $g = e^{2\sigma}\bar{g}$, where $e^{2\sigma}$ is the nowhere vanishing positive function on M. $\bar{g}$ will be called a representative element of the conformal equivalence class. This equivalence relation transforms $Q(M) = Riem(M)$ into the conformal geometry $C(M) = \frac{Riem(M)}{Weyl(M)}$. The Weyl group of conformal rescalings $Weyl(M)$ acts freely on $C(M)$. An element (point) of $Riem(M)$, g is left invariant under the transformation $\bar{g} \to e^{2\chi}\bar{g}$, $\sigma \to \sigma - \chi$. It is important to notice that Weyl invariance corresponds to the freedom of choice of a representative element $\bar{g}$ of the conformal geometry $C(M)$.

The group of diffeomorphisms $Diff(M)$ acts on $Q(M)$ in the following way. Let f is an element of $Diff(M)$, then $g^f = f^* g$,

$$g_{a'b'}(x') = \frac{\partial x^c}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{b'}} g_{cd}(x = f(x')),$$

$$x^a = f^a(x'), x^{a'} = (f^{-1})^a(x). \tag{4}$$

We can write this more concisely, $g^f = \omega_f \omega_f g(f)$, where $\omega_f = df f^{-1}$ is the right-invariant one-form on $Diff(M)$. Once we choose the representative element $\bar{g}$ of the conformal geometry $C(M)$ we can parametrize the space of metrics in the following way

$$g = \left(e^{2\sigma}\bar{g}\right)^f . \tag{5}$$

Starting with an element $\bar{g}$ of the conformal class of metrics $C(M)$ the action of the semidirect product of $Diff(M)$ and $Weyl(M)$ produces a generic metric g from $Q(M) = Riem(M)$. In this way we constructed a natural coordinate system on $Q(M)$, with (σ, f) the natural group coordinates on $Weyl(M)$ and $Diff(M)$. The infinitesimal diffeomorphisms (connected to identity elements of $Diff_0(M)$)

are generated by vector fields ξ. The integrated (finite) elements of $Diff_0(M)$ generated by ξ are denoted by $Exp(\xi)$. The coordinate system on $Q(M)$ is given by a triple $(\sigma, \xi, \bar{g})$ which can be used in the construction of the gravitational measure. The change of variables $g \to (\sigma, \xi, \bar{g})$ requires a Jacobian which we evaluate below after the orthogonal decomposition of a tangent vector $\delta g \in TQ_{|g}$ with respect to the De Witt metric (1) is obtained.

The combined infinitesimal action of $Weyl(M)$ and $Diff(M)$ on $Q(M)$ produces an element δg of the tangent space $TQ(M)$

$$\delta g_{ab} = \left(2\delta\sigma + \frac{2}{d}\nabla_c \xi^c\right) g_{ab} + (L\xi)_{ab}, \tag{6}$$

where $L\xi$ is the trace-free part of the Lie derivative of a metric $\delta_f g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$,

$$(L\xi)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - \frac{2}{d} g_{ab} \nabla_c \xi^c. \tag{7}$$

The operator L maps vectors into trace-free symmetric tensors. The kernel of L, $KerL$, consists of the conformal Killing vectors (CKV). We can define a natural adjoint $L^\dagger$ of the operator L with respect to the De Witt scalar product (2) on $TQ(M)$

$$< h, L\xi > = < L^\dagger h, \xi >,$$

$$\left(L^\dagger h\right)_a = -2\nabla^b h_{ab}, g^{ab} h_{ab} = 0. \tag{8}$$

The adjoint operator $L^\dagger$ maps symmetric trace-free tensors into vectors. It is clear that these elements of the tangent space $TQ(M)$ which are not in the range of L, $Range(L)$, must be in the kernel of $L^\dagger$, $KerL^\dagger$. Those are the transverse trace-free symmetric tensors h_{ab}. Therefore, we have a natural orthogonal splitting of the vector space $TQ(M)_{|g}$

$$\delta g_{ab} = \left(2\delta\sigma + \frac{2}{d}\nabla_c \xi^c\right) g_{ab} + (L\xi)_{ab} + h_{ab}. \tag{9}$$

To find the measure $d\mu(\delta g) = D\delta g$ on the space of metrics we need to choose the representative element $\bar{g} \in C(M)$ of the conformal class of metrics $C(M)$.

This can be done by specifying $\bar{g}$ to lie in a slice $\bar{Y}$ transversal to the orbits of $Weyl(M)$ and $Diff_0(M)$. Such a slice $\bar{Y}$ may be taken to be the Schoen-Yamabe slice defined by the condition of constant Ricci scalar: $\bar{Y} = \{\bar{g}, R(\bar{g}) = const\}$. The recent proof of the Yamabe conjecture [8] for $d = 3, 4, 5$ guarantees the local tranversality of $\bar{Y}$ to the orbits of $Weyl(M)$. For $d > 2$ the Schoen-Yamabe slice $\bar{Y}$ is infinite dimensional; for $d = 2$ its dimension is finite and any metric $\bar{g}$ can be parametrized by the Teichmuller (moduli) parameters t_i, $\bar{g} = \bar{g}(t)$. Formally, we can parametrize a metric $\bar{g} \in \bar{Y}$ by an infinite number of parameters t_i (arbitrary functions), $\bar{g} = \bar{g}(t)$, and tangents to $\bar{Y}$ are symmetric tensors $\bar{\phi}_j$ defined by $\delta\bar{g}(t) = \sum_j \delta t_j \bar{\phi}_j$. The coordinate vectors along $\bar{Y}$ are $\delta\sigma\bar{g}$, $\bar{L}\xi$ and $\bar{\phi}_j$. Once the coordinate system on $Q(M)$ is specified we can easily evaluate the Jacobian at an arbitrary metric g (5). Under the Weyl rescaling $e^{2\sigma}$ the tangent vectors to $\bar{Y}$ transform as: $\phi_j(e^{2\sigma}\bar{g}) = e^{2\sigma}\bar{\phi}_j(\bar{g})$. This choice is a natural one for conformal geometry $C(M)$ and it will be justified later.

It will be useful to introduce a basis of vectors h_j in the kernel of $L^\dagger$, $h_j \in KerL^\dagger$. Using the orthogonal decomposition (9) of δg_{ab} we obtain the following expression for the measure

$$D\delta g = d\mu_g(\delta g) = \frac{det < \phi_j|h_k >_g}{det^{1/2} < h_j|h_k >_g}\Big(det' L^\dagger L\Big)^{1/2} D\sigma D'\xi Dt. \tag{10}$$

In the expression for the measure all the determinants must be properly regularized because all of the matrices or operators act in the infinite dimensional spaces. The reason that determinants of matrices of scalar products appear is because the basis in $KerL^\dagger$ is not tangent to the Schoen-Yamabe slice $\bar{Y}$. We must consider the orthogonal projection of vectors in $T\bar{Y}$ onto $KerL^\dagger$. This way the generalized product of cosines of angles between ϕ_j and h_k multiplies the standard measure on $T\bar{Y}$ (see, e.g., D'Hoker and Phong 1988) [3].

The presence of matrices of scalar products $< \phi_j|h_k >$, $< h_j|h_k >$ makes it important to study their transformation laws under the Weyl rescaling. We assume that the kernel of $L^\dagger$, $KerL^\dagger$, is conformally invariant for any d. Indeed, one can

easily see that the conformal weight w of $h_j \in KerL^\dagger$ defined by $h_j = e^{w\sigma}\bar{h}_j$ is $w = 2 - d$ if $KerL^\dagger$ is conformally invariant

$$\left(L^\dagger h_j\right)_a = e^{(w-2)\sigma}\left[\left(\bar{L}^\dagger \bar{h}_j\right)_a - 2(w+d-2)\bar{h}^b_{ja}\nabla_b\sigma\right]. \qquad (11)$$

The conformal variation of $L^\dagger$ can be easily read off from the last formula

$$\delta L^\dagger = -2\delta\sigma L^\dagger - 2(d-2)\nabla\delta\sigma. \qquad (12)$$

In a similar way we find the conformal weight s of vectors in $KerL$. Consider the transformation $\xi_a = e^{s\sigma}\bar{\xi}_a$, then

$$(L\xi)_{ab} = e^{s\sigma}\left[(\bar{L}\bar{\xi})_{ab} + (s-2)\left(\nabla_a\sigma\bar{\xi}_b + \nabla_b\sigma\bar{\xi}_a - \frac{2}{d}\bar{g}_{ab}\bar{\nabla}^c\bar{\xi}_c\right)\right]. \qquad (13)$$

When $s = 2$, then $KerL$ is conformally invariant. ¿From (13) we find also the conformal variation of L,

$$\delta(L\xi)_{ab} = -2\left(\nabla_a\delta\sigma\xi_b + \nabla_b\delta\sigma\xi_a - \frac{2}{d}g_{ab}\nabla^c\delta\sigma\xi_c\right). \qquad (14)$$

The choice of conformal properties of vectors from $KerL^\dagger$ and $T\bar{Y}$ imply that $<\phi_j|h_k>_g = <\bar{\phi}_j|\bar{h}_k>_{\bar{g}}$. This property of scalar products will prove to be quite important later.

The Jacobian J in the gravitational measure, $J = \left(det' L^\dagger L\right)^{1/2}$ carries important information about how the measure changes along the orbits of the Weyl group of conformal rescalings $Weyl(M)$. As usual, we can find the dependence of J on the conformal factor σ by integrating the conformal anomaly equation. Of course, the situation is more subtle because of the possibility of zero modes of L, which are CKV. We denote $L^\dagger L = \Delta$. This is a vector operator mapping vectors

into vectors which has the following form

$$\Delta_a{}^b = -2\left(\nabla^2\delta_a{}^b + (1-\frac{2}{d})\nabla_a\nabla^b + R_a{}^b\right). \tag{15}$$

We notice in passing that Δ has a diagonal symbol only for $d = 2$, which makes the heat-kernel approach calculation of the conformal anomaly much easier in this case. For $d \neq 2$ the nondiagonal symbol of Δ complicates calculations significantly.

We evaluate the conformal change of lnJ using the Schwinger proper time definition of the functional determinant

$$lnJ = -\frac{1}{2}Tr'\int_{\epsilon}^{\infty}\frac{dt}{t}e^{-t\Delta}, \tag{16}$$

where Tr' denotes the functional trace operation over non-zero modes of Δ. As usual, we calculate first a variation of lnJ under the local change of the conformal factor σ

$$\delta lnJ = \frac{1}{2}Tr'\int_{\epsilon}^{\infty}dt\delta\Delta e^{-t\Delta}. \tag{17}$$

Using the operator identity $Ae^{-BA} = e^{-AB}A$, eqs. (12),(14) and the expression for the variation of Δ, $\delta\Delta = \delta L^\dagger L + L^\dagger\delta L$, we obtain the following formulas

$$Tr'\delta\Delta e^{-t\Delta} = Tr'\left(\delta L^\dagger L e^{-tL^\dagger L}\right) + Tr'\left(L^\dagger\delta L e^{-tL^\dagger L}\right), \tag{18}$$

$$Tr'\delta L^\dagger L e^{-tL^\dagger L} = (d-2)Tr'\delta\sigma LL^\dagger e^{-tLL^\dagger} - dTr'\delta\sigma L^\dagger L e^{-tL^\dagger L}, \tag{19}$$

$$Tr'L^\dagger\delta L e^{-tL^\dagger L} = 2Tr'\delta\sigma LL^\dagger e^{-tLL^\dagger} - 2Tr'\delta\sigma L^\dagger L e^{-tL^\dagger L}. \tag{20}$$

Collecting these results together we obtain a formula which allows for an explicit proper time integration in (17)

$$Tr'\delta\Delta e^{-t\Delta} = (d+2)Tr'\delta\sigma\frac{d}{dt}e^{-tL^\dagger L} - dTr'\delta\sigma\frac{d}{dt}e^{-tLL^\dagger}. \tag{21}$$

Finally, we find the following useful expression for the conformal variation of

$lndet' L^\dagger L$

$$\delta lndet' L^\dagger L = dTr' \delta\sigma e^{-\epsilon L L^\dagger} - (d+2) Tr' \delta\sigma e^{-\epsilon L^\dagger L}. \tag{22}$$

We notice that in all these formulas the trace is taken over nonzero modes. Introducing projectors $P(L)$, $P(L^\dagger)$ on the respective kernels, $KerL$ and $KerL^\dagger$, we can rewrite the last formula in the following way

$$\delta lndet' L^\dagger L = dTr\delta\sigma e^{-\epsilon L L^\dagger} - (d+2) Tr\delta\sigma e^{-\epsilon L^\dagger L} + (d+2) Tr\delta\sigma P(L) - dTr\delta\sigma P(L^\dagger). \tag{23}$$

At this point the previous discussion of conformal weight assignment for vectors from $KerL$ and $KerL^\dagger$ proves quite useful. We find

$$\delta lndet < \xi_j | \xi_k >= (d+2) Tr\delta\sigma P(L), \tag{24}$$

where $\xi_j \in KerL$ and

$$\delta lndet < h_j | h_k >= -dTr\delta\sigma P(L^\dagger). \tag{25}$$

The short time ($\epsilon \to 0^+$) expansion of heat kernels for both operators $L^\dagger L$ and $LL^\dagger$ is required for the calculation of a conformal anomaly for the vector operator Δ. Integrating the conformal anomaly leads to a local effective action in the conformal gauge. We can see that the presence of zero modes in the Jacobian is universal for any dimension d

$$\begin{aligned} \delta ln \left[\frac{det' L^\dagger L}{det < \xi_j | \xi_k > det < h_j | h_k >} \right]^{1/2} &= \frac{1}{2} \left[dTr\delta\sigma e^{-\epsilon L L^\dagger} - (d+2) Tr\delta\sigma e^{-\epsilon L^\dagger L} \right] \\ &= -\delta\Gamma, \end{aligned} \tag{26}$$

where Γ is the effective action for the conformal factor σ, which in the case of two-geometries was shown by Polyakov to be given in terms of the celebrated Liouville action. Our goal is to find the analog of the Liouville action in higher dimensions.

The quantum dynamics of the conformal factor σ in higher dimensions (especially for $d = 4$) is governed by the renormalizable, but nonlinear, effective action $\Gamma[\sigma;\bar{g}]$. In order to find Γ we need to integrate the anomaly equation

$$\delta\Gamma = \frac{1}{2}\left[(d+2)Tr\delta\sigma e^{-\epsilon L^\dagger L} - dTr\delta\sigma e^{-\epsilon LL^\dagger}\right] = \int \sqrt{g}dx\delta\sigma A(g), \qquad (27)$$

where $A(g)$ is the local conformal anomaly. To integrate the anomaly equation (27) we consider a one-parameter family of metrics $g(\tau)$, $\tau \in [0,1]$ interpolating between $\bar{g}$ and $g = e^{2\sigma}\bar{g}$: $g(\tau) = e^{2\tau\sigma}\bar{g}$. Then, $\delta\sigma = d\tau\sigma$, $\sqrt{g(\tau)} = e^{d\tau\sigma}\sqrt{\bar{g}}$. The local anomaly A is given in terms of local curvature invariants for the metric $g(\tau)$. ¿From the transformation properties of the Riemann tensor under the local conformal rescaling we can easily find how the conformal anomaly A depends on τ and σ. The effective action Γ is given by a simple integral over τ

$$\Gamma[\sigma;\bar{g}] = \int_0^1 d\tau \int dx\sqrt{\bar{g}}\sigma e^{d\tau\sigma}A[\tau,\sigma;\bar{g}]. \qquad (28)$$

Now we are in a position to express the ratio of two determinants evaluated at two metrics $\bar{g}$ and $g = e^{2\sigma}\bar{g}$ in terms of the effective action Γ

$$\left[\frac{det' L^\dagger L}{det < \xi_j|\xi_k > det < h_j|h_k >}\right]_g^{1/2} = e^{-\Gamma[\sigma;\bar{g}]}\left[\frac{det' \bar{L}^\dagger \bar{L}}{det < \bar{\xi}_j|\bar{\xi}_k > det < \bar{h}_j|\bar{h}_k >}\right]_{\bar{g}}^{1/2}. \qquad (29)$$

The dependence of the gravitational measure on the conformal factor σ is obtained using eqs. (10) and (29)

$$\begin{aligned} d\mu(e^{2\sigma}\bar{g}) = e^{-\Gamma[\sigma;\bar{g}]}\left(det'\bar{L}^\dagger\bar{L}\right)_{\bar{g}}^{1/2} \left[\frac{det < \xi_j|\xi_k >_g det < h_j|h_k >_g}{det < \bar{\xi}_j|\bar{\xi}_k >_{\bar{g}} det < \bar{h}_j|\bar{h}_k >_{\bar{g}}}\right]^{1/2} \times \\ \times \frac{det < \phi_j|h_k >_g}{det^{1/2} < h_j|h_k >_g} D\sigma D'\xi Dt \quad . \end{aligned} \qquad (30)$$

Using the conformal properties of ϕ_j and h_j we finally arrive at the nicely factorized

expression for the gravitational measure

$$d\mu(e^{2\sigma\bar{g}}) = e^{-\Gamma[\sigma;\bar{g}]} \left(\frac{det'\bar{L}^\dagger \bar{L}}{det < \bar{\xi}_j|\bar{\xi}_k >} \right)^{1/2}_{\bar{g}} dVol(Diff_0) D\sigma d\mu_{\bar{g}}(t), \tag{31}$$

where

$$d\mu_{\bar{g}}(t) = \frac{det < \bar{\phi}_j|\bar{h}_k >_{\bar{g}}}{det^{1/2} < \bar{h}_j|\bar{h}_k >_{\bar{g}}} Dt, \tag{32}$$

is the invariant, σ independent, measure on the Schoen-Yamabe slice $\bar{Y}$, and $dVol(Diff_0) = det^{1/2} < \xi_j|\xi_k >_g D'\xi$ is the measure on the group of difeomorphisms connected to identity $Diff_0$. The first factor in $dVol(Diff_0)$ is the volume form on the conformal group generated by CKV's $\xi_j \in KerL$. After dividing out the volume of the group of general covariance we are left with the measure on the Schoen-Yamabe slice $\bar{Y}$ and the induced effective action for the conformal factor σ.

The short-time heat kernel expansion for the vector and tensor operators $L^\dagger L$ and $LL^\dagger$ on manifolds without boundary contains local curvature invariants of an appropriate dimension (depending on d). The cases of interest for us are three- and four-geometries ($d = 3$, $d = 4$) for which we know the general form of the conformal anomaly. The only thing left is to determine exactly the numerical coefficients in front of different curvature invariants. In a separate paper we will discuss the exact form of a conformal anomaly for four-geometries. In the case of three-geometries the situation is much simpler. On a manifold without boundary the short-time heat hernel expansion has the universal form

$$Tre^{-t\Delta} = (4\pi t)^{-\frac{d}{2}} \left(a_0 + a_1 t + a_2 t^2 + ...\right), \tag{33}$$

where $a_k[g]$ are integrals of local curvature invariants. In particular, when d is odd the conformal anomaly $a_{\frac{d}{2}}$ vanishes automatically. It can be present only when a manifold has a boundary, in which case the expansion (33) will have terms with half-integer powers of t. Those are the boundary terms which depend on the

geometry of a boundary. In the case of three-manifold with a boundary the heat kernel expansion has a form

$$Tre^{-t\Delta} = (4\pi t)^{-\frac{3}{2}} \left(a_0 + b_{\frac{1}{2}} t^{\frac{1}{2}} + a_1 t + b_{\frac{3}{2}} t^{\frac{3}{2}} + ... \right), \tag{34}$$

where $b_{\frac{1}{2}}$ and $b_{\frac{3}{2}}$ are the surface terms. a_0 is the cosmological (volume) term, a_1 is the Einstein-Hilbert term given by the integral of the Ricci scalar. In general, $b_{\frac{1}{2}}$ is the $2D$ cosmological term on the boundary and $b_{\frac{3}{2}}$ is the $2D$ Einstein term. The conformal anomaly in $d = 3$ is given completely by the surface term. From (27) we can find the effective action which will have the volume and surface (boundary) contributions. The volume contribution, which is divergent, comes from integrating the Einstein-Hilbert action. Obviously, this anomaly (in any dimension) comes from the local action, and by adding a local counterterm which is the Einstein-Hilbert action we can cancel it. We conclude that the divergent volume part of the effective action Γ renormalizes the cosmological and Newton constants. In other words, the Jacobian in the gravitational measure for three-geometries induces $3D$ gravity. By fine tuning the bare cosmological and Newton constants we can completely cancel the volume contribution to the effective action. What is left is the boundary contribution. But the boundary contribution to the conformal anomaly is the Einstein term, the Ricci scalar. Integrating this conformal anomaly we obtain the celebrated Liouville action on the boundary of a three-manifold. The total effective action Γ has a form

$$\Gamma = \frac{1}{16\pi G} \int d^3x (R - 2\Lambda) + cI_L, \tag{35}$$

where I_L is the Liouville action depending on the metric on the boundary. We have seen that a quite general argument shows the presence of Liouville action in $3D$ gravity. In principle other terms involving the extrinsic curvature K_{ij} could be present in the conformal anomaly. These are terms like $K_{ij}K^{ij}$, K^2, where $K = K_i{}^i$, but by the Gauss-Coddazi equation only one of these terms could appear. However, such terms cannot appear because they would spoil the naive

factorization of the measure. For any three-manifold with a boundary we can take the double of it and produce a manifold without a boundary. Now, the measure on a manifold without a boundary does not contain (surface) boundary contribution to the Jacobian. Naive factorization of the measure on the doubled manifold means that we have to take the product of two measures on a manifold with boundary and integrate over metrics on the dividing boundary. This requires the measure on a boundary which is the known measure for two-geometries. The measure over two-geometries has a Jacobian which is $exp(-26I_L)$, where I_L is the Liouville action [2]. The dependence of the measure on a three-manifold with boundary on the geometry of a boundary must be of the same form, and cannot depend on the extrinsic curvature of a boundary.

We may ask a question, What is the meaning of the Liouville action in the measure for a three-manifold with a boundary ? Adopting the point of view of induced gravity, which proves to be quite useful for $d = 2$ case, we can argue that by integrating over all metrics on a three-manifold with boundary with a fixed two-metric on a boundary (and fine tuning coefficients in the Einstein-Hilbert action to zero) we obtain the Hartle-Hawking wave function for the $3D$ Universe. The difference between $2D$ and $3D$ induced gravity is that in $d = 2$ case the quantum measure induces the effective action, and in the $d = 3$ case it is the Hartle-Hawking wave function which is effectively induced. Of course, one may argue that we have fine tuned coefficients in such a way that the Einstein-Hilbert action vanishes and we are not calculating the HH wave function for $3D$ gravity. Also, we get an infinite factor in the integral which is the volume of the space of metrics but this is the usual factor which can be taken care off by proper normalization of the HH wave function. We consider this interpretation of the appearance of the Liouville action in the induced HH wave function as a very attractive possibility. Indeed, it is well known since the work of Banks et al. [9] that the wave function for $3D$ gravity which is the solution of the Wheeler-DeWitt equation in the gaussian approximation is given in terms of the Liouville action evaluated to the same gaussian order. It is plaussible, therefore, that the exact HH wave function (not only the induced one)

is given in terms of the Liouville action.

As a side remark we would like to describe how the induced HH wave function appears in the Chern-Simons gravity (or in any Chern-Simons $3D$ model with any gauge group). Witten has shown that on-shell, i.e., classically, the Einstein-Hilbert metric formulation of $3D$ gravity and the Chern-Simons gravity which is the $ISO(2,1)$ gauge theory are equivalent [10]. Now, the Chern-Simons gravity is (super)renormalizable, but naively the Einstein-Hilbert gravity is nonrenormalizable. The situation here is quite similar to that which occurs in string theory. In the Polyakov formulation the string model action is classically equivalent to the Nambu-Goto form of the string action. The Nambu-Goto action is nonrenormalizable but in the Polyakov formulation the model is renormalizable. The partition function for the Chern-Simons gravity is given in terms of the Ray-Singer analytic torsion invariant and does not depend on the $3D$ metric which was chosen to gauge fix the action. This happens because the contributions to the effective action, which may depend on the metric, from the ghosts and the gauge field cancel exactly. In any case, the possible (volume) contribution would have had the form of the Einstein-Hilbert action if it were not vanishing exactly. What happens when we consider the partition function of the Chern-Simons gauge theory for the three-manifold with a boundary ? In this case it is easy to convince ourselves that the dependence on the gauge-fixing metric is nontrivial because the boundary contributions to the conformal anomaly from ghosts and gauge fields do not cancel each other. In fact, the boundary contribution to the partition function is given by $exp(cI_L)$, where c is the central charge of the corresponding conformal field theory which “lives” on the boundary. In this sense the topological character of the Chern-Simons gravity (and other CS models with different gauge groups) is broken by gravitational conformal anomaly. Boundaries are responsible for breaking the topological invariance of CS model. Once again we may adopt a point of view that in analogy to $2D$ case, where for each conformal field theory we get induced gravity, any Chern-Simons $3D$ gauge model leads to induced $3D$ gravity. The only difference is that in the $3D$ CS case we get the induced HH wave function of the

Universe, rather than the induced action.

We conclude with a short discussion of the main results of this paper. The main new result appears to be the construction of the universal (for any dimension d) quantum gravitational measure which generalizes the Polyakov measure for two-geometries. The difference between odd and even dimensions was pointed out, and the special example of the quantum gravitational measure for three-geometries shows the important role of boundaries in odd dimensions. In particular, we argued that the metric dependence of the measure for three-geomerties is given by the Einstein-Hilbert action and the boundary contribution which is given in terms of the Liouville action. This quite mysterious fenomenon of appearance of the Liouville action in quantum $3D$ gravity, encountered earlier by Banks et al., is interpreted in terms of the induced HH wave function (rather than induced action). We compared this situation to that which occurs in the case of the Chern-Simons gravity where we get an exact induced HH wave function for $3D$ gravity as a result of conformal anomaly. The more complicated case of the measure for quantum four-geoemtries requires an exact calculation of the effective action for the conformal factor. This will be discussed in the forthcoming paper.

I would like to thank K. Aoki, E. D'Hoker, J. B. Hartle, E. Mottola, T. Tomboulis for discussions on the subject of the gravitational measure, R. Schoen for explaining to me some nuances of the Yamabe problem, and R. Peccei for continuing support. I also enjoyed discussion with T. Banks and L. Susskind about the WDW wave function in 3D gravity.

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